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Non-renormalisation conditions in type II string theory and maximal supergravity

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ABSTRACT: This paper considers general features of the derivative expansion of Feynman diagram contributions to the four-graviton scattering amplitude in eleven-dimensional supergravity compactified on a two-torus. These are translated into statements about interactions of the form $D^{2k}R^4$ in type II superstring theories, assuming the standard M-theory/string theory duality relationships, which provide powerful constraints on the effective interactions. In the ten-dimensional IIA limit we find that there can be no perturbative contributions beyond k string loops (for $k > 0$). Furthermore, the genus $h = k$ contributions are determined exactly by the one-loop eleven-dimensional supergravity amplitude for all values of k . A plausible interpretation of these observations is that the sum of h -loop Feynman diagrams of maximally extended supergravity is less divergent than might be expected and could be ultraviolet finite in dimensions $d < 4 + 6/h$ – the same bound as for $N = 4$ Yang-Mills.

KEYWORDS: String Duality, M-Theory, Extended Supersymmetry.

Contents

1. Introduction and overview	1
2. Eleven-dimensional supergravity compactified on a two-torus	4
2.1 Comments on the structure of eleven-dimensional Feynman diagrams	5
2.2 Lessons from known higher derivative terms in the IIA and IIB actions	7
2.3 The L -loop four-graviton amplitude compactified on a two-torus	9
2.4 Relation to type II string theories	11
3. Higher derivative interactions in type IIB	14
3.1 Terms that are finite as $r_B \rightarrow \infty$	14
3.2 Terms that diverge as $r_B \rightarrow \infty$	14
3.2.1 Terms that sum to the ten-dimensional $s \ln(-\alpha' s)$ threshold	16
3.2.2 Terms that sum to the ten-dimensional $\alpha'^3 s^4 \ln(-\alpha' s)$ threshold at genus one and two	17
4. Higher derivative interactions in type IIA	18
4.1 Terms that are finite as $r_A \rightarrow \infty$ — non-renormalisation conditions	19
4.2 Terms that diverge as $r_A \rightarrow \infty$	20
4.2.1 Terms that sum to the ten-dimensional $s \ln(-\alpha' s)$ threshold	21
4.2.2 Terms that sum to the ten-dimensional $\alpha'^3 s^4 \ln(-\alpha' s)$ threshold at genus one and two	21
4.3 Lifting to eleven dimensions	22
5. Reduction to lower dimensions	23
A. Properties of tree-level and one-loop string theory	27
A.1 Tree-level four graviton scattering in type II string theory	27
A.2 One-loop four-graviton scattering in type II string theory	28
B. Ten-dimensional thresholds from nine dimensions	28

1. Introduction and overview

Although the complete non-perturbative description of string theory remains unfathomable, a variety of non-perturbative features of the derivative expansion of the string theory effective action have been deduced over the years. This expresses the low energy dynamics in terms of the massless fields of the theory after the massive modes have been integrated out. Considerations have typically been limited to a few higher-derivative terms (i.e., a

few powers of α') describing a sub-sector of the theory. A most interesting open question is to what extent supersymmetry, together with various dualities might constrain the non-perturbative structure of the terms in the effective action. In this paper we will uncover some systematic properties of the IIA and IIB string effective actions that follow from general features of the Feynman diagrams of eleven-dimensional supergravity combined with constraints that enforce the duality relationships between M-theory and string theory at the quantum level.

At the classical level, S -duality identifies eleven-dimensional supergravity compactified on a two-torus with classical type IIA or type IIB supergravity compactified to nine dimensions on a circle [1–3]. This identification may be extended to the quantum theory by considering the compactification of loop diagrams of eleven-dimensional supergravity [4–7]. We will here argue that if one assumes that the duality conditions continue to hold at higher orders, strong constraints are imposed on the possible higher-genus corrections to higher-derivative terms in the ten-dimensional type II effective actions. We will find, in particular, that the IIA supergravity genus expansion must satisfy a strong non-renormalisation condition that makes it much less ultraviolet divergent than would otherwise be suspected. In fact, our work indicates that, after reduction to d dimensions, the sum of all diagrams with h loops is finite in dimensions $d < 4 + 6/h$.

There is obviously much more to M-theory than the Feynman diagrams of eleven-dimensional supergravity, which are based on the dynamics of the superparticle and do not explicitly include deeper aspects of M-theory associated, for example, with M -branes. The non-renormalisability of supergravity perturbation theory is a symptom that it does not adequately take short-distance physics into account. Our procedure in the following will be to introduce an ultraviolet cut-off and subtract divergent terms with counterterms that have unknown coefficients that parameterize our lack of knowledge of short-distance effects that might be determined from first principles in a more fundamental formulation of M-theory. We will then impose conditions on the expressions that are required by duality relationships for consistency with known properties of string theory.

The structure of the paper is as follows. In section 2 we will consider the general structure of the four-graviton amplitude obtained by compactifying the Feynman diagrams of eleven-dimensional supergravity on a two-torus of volume \mathcal{V} and complex structure Ω . Our considerations will be restricted to properties of L -loop Feynman diagrams on a two-torus that do not depend on detailed analysis of the individual diagrams. We are imagining introducing a momentum cut-off Λ to the loop momenta in such diagrams, which contribute to interactions starting from terms of the form $D^{2\beta_L} R^4$, where β_L is an integer. As we will briefly review in section 2.1, summing all the Feynman diagrams that contribute to the four-graviton amplitude is known to result in very simple expressions for $L = 1$ with $\beta_1 = 0$, and $L = 2$ with $\beta_2 = 2$. The full dilaton dependence of several low-order terms in the type IIA and IIB string theory derivative expansions have been obtained in explicit detail from the toroidally compactified $L = 1$ and $L = 2$ amplitudes, making use of the standard duality relation between M-theory and type II string theory. These examples, which provide lessons for the rest of the paper, are reviewed in section 2.2.

General features of four-graviton L -loop Feynman diagrams compactified on a torus

will be described in section 2.3. The superficial degree of divergence of the L -loop diagram is $\Lambda^{9L-6-2\beta_L}$ (where $9L-6-2\beta_L > 0$). However, this power is reduced by Λ^{-w} in subdivergent terms that have compensating factors of $\mathcal{V}^{-w/2}$ with integer $w > 0$. Furthermore, the four-graviton amplitude has an expansion in powers of the Mandelstam invariants of the form¹ $(\mathcal{V}S)^v$, corresponding to an infinite series of derivatives, $D^{2k}R^4$, where $k = v + \beta_L$. Invariance under large diffeomorphisms of the torus implies that dependence on the complex structure Ω is necessarily encoded in a $\text{SL}(2, Z)$ -invariant function.

In subsection 2.4 the compactified eleven-dimensional action is translated into string variables. The well-known duality relations between M-theory and string theory are used to make appropriate identifications of the parameters \mathcal{V} and Ω with the moduli of IIA or IIB string theory compactified on a circle (of radius r_A and $r_B = 1/r_A$, respectively). This leads to a generic description of the coefficients of the $D^{2k}R^4$ interactions in which there are powers of the radius, r_A^{1+p} or r_B^{1+p} (where p depends on w and k), and dependence on the coupling, e^ϕ (where ϕ is the IIA or IIB dilaton) and Ramond-Ramond pseudoscalar (for IIB) or vector (for IIA). Imposing these duality rules leads to powerful constraints on quantities that are undetermined by our eleven-dimensional supergravity starting point. For example, we need to stipulate that the string expansions are in even powers of the string coupling constant $e^{2\phi}$ and that the most singular term is the tree-level term of order $e^{-2\phi}$. When reinterpreted in terms of the eleven-dimensional theory these conditions severely restrict the possible terms in the effective action.

The treatment of the type IIA and type IIB cases is rather different. In section 3 we consider the IIB parametrization, where the complex structure Ω is identified with the complex coupling constant and the coupling constant dependence of $D^{2k}R^4$ is encoded in modular functions, $f_{(q,k)}(\Omega, \bar{\Omega})$ (where $q = w/2 - v$). Terms that are finite in the ten-dimensional limit have $p = 0$ and the value of q is determined by k . Little can be said about the precise form of the modular function $f_{(q,k)}$ without detailed calculations, such as those that have been determined previously for $k = 0, 1, 2, 3$, which are solutions of Poisson equations in moduli space. There are also infinite classes of terms with $p > 0$ that diverge in the ten-dimensional limit ($r_B \rightarrow \infty$). We argue that these terms arise from the low energy expansion of multi-particle threshold singularities for massive Kaluza-Klein modes. These modes condense to zero mass in the ten-dimensional limit and the series of positive powers of r_B must resum to generate the logarithmic threshold terms that are known to satisfy the ten-dimensional unitarity constraints. This effect was discussed for the IIA $L = 1$ case in [7]. We will demonstrate how this works for the first few threshold terms in the IIB case.

In the type IIA case considered in section 4 the ten-dimensional limit appears in a very different manner. Unlike the IIB case, the value of q for terms that are finite in ten dimensions depends on both k and the string loop genus h . Different orders in the string loop expansion arise from different values of L . This leads to some surprisingly strong constraints. In particular, in the ten-dimensional limit we find that there are no

¹We are using capital letters S, T, U for the eleven-dimensional Mandelstam invariants and lower-case letters for the string theory invariants.

contributions to $D^{2k}R^4$ beyond $h = k$ loops ($h \geq 1$). Furthermore, the $h = k$ contributions are given exactly by the compactification of one-loop ($L = 1$) supergravity for all values of $k > 0$. Terms with $h < k$ can get contributions from arbitrary large values of L .

As mentioned earlier our procedure is based on assuming the familiar M-theory/string theory duality relations, which imposes constraints that are not apparent in ordinary supergravity. In section 4.3 we will see how the consistency of this procedure restricts the kind of terms that arise when the theory is lifted back to eleven dimensions. Since these arguments are based largely on power counting and on the relation between type II string theory and M-theory parameters they suggest that similar non-renormalisation conditions might also apply the many other interactions of the same dimension as $D^{2k}R^4$.

The non-renormalisation conditions of section 4.1 provide very powerful restrictions on the ultraviolet behaviour of low energy IIA supergravity. The consequences of these restrictions will be discussed in detail in section 5. There are tantalizing hints that the consistency of the picture presented requires the supergravity theory to be considerably more finite than is evident from the superficial degree of divergence of individual Feynman diagrams. Our results imply that at each successive loop there is an extra factor of D^2 . In other words, the sum of all diagrams with h loops has a factor of $D^{2h}R^4$ so that the degree of divergence is reduced to that of φ^4 scalar field theory or electromagnetism coupled to φ^3 scalar fields. If we assume that this feature survives after compactification this is just what is needed in order for the Feynman diagrams of $N = 8$ supergravity in four dimensions to be ultraviolet finite. However, care should be taken in interpreting this low energy limit of string theory since it is one in which infinite towers of ‘non-perturbative’ states become massless. This may cast doubt on the usefulness of the perturbative supergravity approximation, which only includes the massless perturbative states (similar observations have also been made by H. Ooguri and J.H. Schwarz (private communication)).

2. Eleven-dimensional supergravity compactified on a two-torus

The description of the effective low-energy theory obtained from eleven dimensions depends sensitively on the spectrum of massless fields, which are responsible for branch cuts in amplitudes, and hence for nonlocal terms in the action. Since the spectrum of massless fields is generally a function of the moduli, the form of the effective action is different in different patches of moduli space. Here we will be concerned with asymptotically flat nine, ten and eleven-dimensional space-time. Thus, starting in nine dimensions, infinite numbers of Kaluza-Klein modes become massless at the boundaries of moduli space where one or two compact dimensions decompactify. This condensate generates a change in the threshold behaviour of amplitudes or, equivalently, generates non-local terms in the higher-dimensional actions. Although we will often talk in terms of the effective action, this can be viewed as a useful shorthand for encoding the properties of the Feynman diagrams that we are considering. Properties of the expansion of the amplitude in terms of analytic and non-analytic functions of Mandelstam invariants translate into properties of local and nonlocal terms in the effective action.

The derivative expansion of the local terms in the effective action is an expansion in powers of $\alpha' = l_s^2$ (where l_s is the string length scale) of the form

$$\alpha'^4 S = S^{(0)} + \alpha'^3 S^{(3)} + \alpha'^5 S^{(5)} + \alpha'^6 S^{(6)} + \dots, \quad (2.1)$$

where $S^{(n)}$ denotes the contribution to the action with dimension $2n - 8$. The known properties of the interactions contained in $S^{(n)}$ are based on a combination of perturbative string theory and the constraints imposed by dualities and supersymmetry. Our explicit considerations concern the expansion of the four-graviton amplitude and so they will be restricted to the linearized approximation to local interactions of the form

$$R^4, D^4 R^4, D^6 R^4, \dots, D^{2k} R^4, \dots, \quad (2.2)$$

although many of our general considerations could also apply to other interactions of the same dimension, such as R^{k+4} . The expression $D^{2k} R^4$ is shorthand for the particular contractions of tensor indices on the linearized curvature tensors and derivatives, which will not be relevant in this paper.² Any possible $D^2 R^4$ interaction vanishes on shell and does not contribute to the four-graviton amplitude.

2.1 Comments on the structure of eleven-dimensional Feynman diagrams

We will begin by summarizing what is known in general about the structure of the Feynman diagrams of eleven-dimensional supergravity.

It has been known for a long time that the sum of the one-loop ($L = 1$) contributions to four-graviton scattering has the structure of an eleven-dimensional φ^3 scalar field theory box diagram multiplying R^4 (see figure 1). In space-time dimensions $d \geq 8$ the box is divergent and so it needs to be regularized. It is cubically divergent (Λ^3) in eleven dimensions. We will subtract this divergence by introducing a counterterm as illustrated in the figure. The fact that all the non-trivial dependence on the external momenta is contained in a scalar field theory box diagram makes this amplitude very easy to evaluate.

The two-loop ($L = 2$) contributions to four-graviton scattering also sum together in a remarkably simple manner. The results of [8] show that the sum has an explicit overall factor of the form $S^2 R^4$ multiplying two particular two-loop Feynman ladder diagrams of φ^3 scalar field theory, together with their T and U channel symmetrization (see figure 2). This means that $D^4 R^4$ is the leading low energy contribution of the two-loop supergravity amplitude and the two-loop scalar amplitude is divergent in dimensions $d \geq 7$. In addition to a primitive divergence that behaves as Λ^8 when $d = 11$ (for a momentum cut-off Λ), it also has one-loop sub-divergences that are proportional to Λ^3 if $d = 11$. The contribution with a single black blob in figure 2 is a diagram with the insertion of the one-loop counterterm, which subtracts a sub-divergence. The double-blob indicates the counterterm for the primitive divergence. Whereas the Λ^3 subdivergent contributions fit in perfectly with string theory expectations (see section 2.2), the two-loop Λ^8 contribution is inconsistent

²For example, at tree-level these contractions can be extracted from the expressions in appendix A.1. For example, the $D^4 R^4$ term has the form $t_8 t_8 R^2 D^\mu D_\mu R^2$, where t_8 is the familiar eighth-rank tensor with indices that contract into pairs of indices from each of the R 's.

$$R^4 \text{ (box diagram)} + R^4 \text{ (four-point vertex)} \Lambda^3$$

Figure 1: The scalar field theory box diagram and the counter-term that subtracts the Λ^3 divergence.

$$D^4 R^4 \text{ (double-box)} + D^4 R^4 \text{ (crossed-box)} + D^4 R^4 \text{ (triangle)} \Lambda^3 + D^4 R^4 \text{ (four-point vertex)} \Lambda^8$$

Figure 2: The two-loop four-graviton amplitude in eleven dimensions is given by the sum of scalar field theory double-box diagrams and counterterms that subtract the primitive divergence and subdivergences.

with string perturbation theory, so the coefficient of its counterterm must be chosen so that the contribution of this term vanishes [7].

There has been no complete analysis of properties of the sums of diagrams for loop diagrams with $L > 2$. In particular, it is not known whether extra factors of S , T and U factor out of the higher-loop amplitudes. What is known is that there are at least two powers of the invariants in the prefactor for the sum of loops for any $L \geq 2$, so in the following we will assume that the sum of L -loop Feynman diagrams has an overall factor of $D^{2\beta_L} R^4$, where $\beta_L \geq 2$. This allows for the possibility that extra derivatives might be extracted at L -loops although our arguments do not assume this.

In the absence of a simple description of the sum of Feynman diagrams when $L > 2$ we will not be able to evaluate the higher loop effects explicitly and our discussion will be

based on general issues, making extensive use of dimensional analysis. Wherever a power of the cut-off appears it signifies divergences that are canceled by counterterms and the result is finite or zero. In the following, when we use the symbol Λ it will usually be with the understanding that the cut-off has been canceled by a counterterm so that Λ^m denotes a finite but undetermined constant with dimensions l_P^{-m} , where l_P is the eleven-dimensional Planck distance.

2.2 Lessons from known higher derivative terms in the IIA and IIB actions

In order to illustrate the general structure we will first review in more detail the explicit description of certain string theory effective interactions that follow from one-loop ($L = 1$) and two-loop ($L = 2$) eleven-dimensional supergravity outlined in the last subsection. This has lead to an understanding of the dilaton dependence of interactions up to $\alpha'^2 D^6 R^4$, which we will review in this subsection.

The first term beyond the classical supergravity action, $S^{(0)}$ (the supersymmetric completion of the $O(\alpha'^{-4})$ Einstein-Hilbert action) is $S^{(3)}$, which includes the well-known R^4 interaction. In the IIB theory this is given by [9]

$$S^{(3)} \sim \int d^{10}x \sqrt{-g} \Omega_2^{\frac{1}{2}} Z_{\frac{3}{2}}(\Omega, \bar{\Omega}) R^4, \quad (2.3)$$

where $\Omega = \Omega_1 + i\Omega_2$ is the complex scalar field of the type IIB theory ($\Omega_2 = \exp(-\phi^B)$ is the type IIB coupling constant) and $Z_{3/2}(\Omega, \bar{\Omega})$ is a $SL(2, Z)$ -invariant function. Various arguments have established that $Z_{3/2}$ is a non-holomorphic Eisenstein series, which is the $s = 3/2$ case of the more general series,

$$Z_s = \sum_{(m,n) \neq (0,0)} \frac{\Omega_2^s}{|m + n\Omega|^{2s}}. \quad (2.4)$$

In addition to (2.3) there are many other interactions of the same dimension that are related to R^4 by the classical supersymmetries. These in general involve combinations of fields that transform as $(-w, w)$ forms under the $SL(2, Z)$ duality group [10] (where the notation indicates holomorphic and anti-holomorphic weights). The coefficient functions that generalize $Z_{3/2}$ are $(w, -w)$ -forms, $Z_{3/2}^{(w, -w)}$. The full set of such interactions has not been enumerated, although they are known in linearized approximation. The Eisenstein series, Z_s is the unique solution of a Laplace eigenvalue equation on the fundamental domain of $SL(2, Z)$ with a polynomial growth in Ω_2 at infinity,

$$\Delta_\Omega Z_s = s(s-1)Z_s. \quad (2.5)$$

Expanding (2.4) for large Ω_2 (i.e., for small string coupling) gives

$$\begin{aligned} Z_s(\Omega, \bar{\Omega}) &= 2\zeta(2s)\Omega_2^s + 2\sqrt{\pi}\Omega_2^{1-s} \frac{\Gamma(s - \frac{1}{2})\zeta(2s-1)}{\Gamma(s)} \\ &+ \frac{2\pi^s}{\Gamma(s)} \sum_{k \neq 0} \mu(k, s) e^{-2\pi(|k|\Omega_2 - ik\Omega_1)} |k|^{s-1} \left(1 + \frac{s(s-1)}{4\pi|k|\Omega_2} + \dots \right), \end{aligned} \quad (2.6)$$

where the last term comes from the asymptotic expansion of a modified Bessel function and $\mu(k, m) = \sum_{d|k} 1/d^{2m-1}$. This expression has two power-behaved terms, which are identified with perturbative string theory tree-level and l -loop terms, as well as an infinite number of non-perturbative exponentially suppressed D -instanton contributions. This demonstrates, for example, the perturbative non-renormalisation of R^4 in the IIB theory beyond one string loop. The coefficients of the tree-level and one-loop terms coincide with the known values found directly from perturbative string theory, which are summarized in appendix A.1 and A.2.³ The IIA theory has no non-trivial duality symmetry. Its action has the same perturbative terms as in the IIB case but there are no D -instantons, and hence no non-perturbative contributions.

These properties of the R^4 term can be deduced by evaluating the one-loop contribution to four-graviton scattering in eleven-dimensional supergravity compactified to nine dimensions on a two-torus of volume \mathcal{V} and complex structure Ω . The amplitude can be expressed as a sum of windings of the loop around the two cycles of the torus, in which case the ultraviolet divergence is entirely in the zero winding sector. The duality properties of M-theory imply that the IIB theory is obtained in the limit in which the volume of the two-torus vanishes, in which case the coefficient of the ultra-violet divergence also vanishes, giving a finite result proportional to $Z_{3/2}(\Omega, \bar{\Omega})$. Furthermore, the IIA theory arises from the limit in which the torus decompactifies to a circle. In that case, there is a finite contribution proportional to $1/R_{11}^3$ from the sum of terms in which the loop has non-zero winding around the circle, which is identified with the string theory tree-level contribution. The ultra-violet divergence is contained entirely in the zero winding term and its coefficient is proportional to Λ^3 (where Λ is a momentum cut-off), which needs to be canceled by adding a counterterm (see figure 1). Since the zero winding sector is independent of R_{11} it corresponds to one loop ($h = 1$) in the IIA string theory. As shown in [4], the precise renormalised coefficient is fixed by imposing the fact that the IIA and IIB string theories are known to have identical four-graviton one-loop amplitudes.⁴ The value of the the IIB one-loop coefficient contained in $Z_{3/2}$ therefore determines the one-loop counterterm to take the value $2\pi^2/3 - \hat{\Lambda}^3$ where $\hat{\Lambda} = l_P \Lambda$.

The next order of the derivative expansion that contributes to four-graviton scattering in the IIB theory is an interaction of the form [7]

$$S^{(5)} \sim \int d^{10}x \sqrt{-g} \Omega_2^{-\frac{1}{2}} Z_{\frac{5}{2}}(\Omega, \bar{\Omega}) D^4 R^4. \quad (2.7)$$

The function $Z_{5/2}$ contains perturbative tree-level and two-loop terms, which have coefficients that agree precisely with those calculated in perturbative string theory, and there

³The expansion of the one-loop amplitude up to the coefficient of $D^6 R^4$ was considered in [11]. Recently, we have extended this to include the coefficients of the $D^8 R^4$, $D^{10} R^4$ and $D^{12} R^4$ interactions [12].

⁴The equality of IIA and IIB four-graviton perturbative contributions persists to at least four loops. The difference between perturbative contributions in IIA and IIB string theories lies in the sign of the odd-odd spin structures, which are associated with the presence of $\epsilon^{\mu_0 \dots \mu_9}$ tensors in both the left-moving and right-moving sectors. The indices on these tensors can only be saturated when there are three or more picture-changing operators in the left-moving and right-moving sectors, but this requires at least three loops. However, at three loops it is also possible to show that the amplitudes must be equal, and there is a strong argument for their equality at four loops [13].

are no further perturbative terms. The fact that there is no one-loop $D^4 R^4$ contribution is also consistent with string perturbation theory [11]. In this case the properties of the $D^4 R^4$ interaction can be deduced by considering the two-loop contribution to four-graviton scattering in eleven-dimensional supergravity compactified to nine dimensions on a two-torus [7]. There is now a pair of winding numbers associated with each loop. The $D^4 R^4$ contribution comes from a sector in which one loop has zero winding number, which gives a one-loop subdivergence proportional to Λ^3 (see figure 2). Inserting the same one-loop counterterm as was used in the one-loop R^4 calculation leads to (2.7). As before, the IIA theory is obtained by considering compactification on a circle. The only non-zero contributions come from the power-behaved terms, in this case corresponding to tree-level and two-loop string perturbation theory, and are equal to those of the IIB case.

At the next order in the α' expansion the four-graviton amplitude displays significant new features. This is indicated by the tree-level coefficient which is proportional to $\zeta(3)^2$ (as reviewed in appendix A.1). As shown in [5] the term in the effective IIB action that contributes to four-graviton scattering has the form (in string frame)

$$S^{(6)} \sim \int d^{10}x \sqrt{-g} \Omega_2^{-1} \mathcal{E}_{(\frac{3}{2}, \frac{3}{2})}(\Omega, \bar{\Omega}) D^6 R^4, \quad (2.8)$$

which was obtained by expanding the two-loop eleven-dimensional supergravity amplitude to the first non-leading order in Mandelstam invariants. This $D^6 R^4$ contribution arises as a finite contribution to the two-loop amplitude since it is given by a sum over non-zero windings of the two loops around both the cycles of the two-torus. The modular function $\mathcal{E}_{(3/2, 3/2)}(\Omega, \bar{\Omega})$ satisfies the Poisson equation

$$\Delta \mathcal{E}_{(\frac{3}{2}, \frac{3}{2})} = 12 \mathcal{E}_{(\frac{3}{2}, \frac{3}{2})} - 6 Z_{\frac{3}{2}} \bar{Z}_{\frac{3}{2}}. \quad (2.9)$$

The inhomogeneous source term makes this equation quite different from the Laplace eigenfunction equation (2.5). Its structure was argued in [5] to follow, at least qualitatively, from the constraints of supersymmetry. In this case the expansion of $\mathcal{E}_{(3/2, 3/2)}(\Omega, \bar{\Omega})$ contains tree-level, one-loop, two-loop and three-loop perturbative string theory terms, as well as infinite series of D -instanton and double D -instanton terms. The tree-level and one-loop coefficients precisely reproduce those known from string calculations. The two-loop coefficient has not yet been extracted from the expression for the two-loop string amplitude in [14–16] so it remains a ‘prediction’. Rather remarkably, the type IIB three-loop coefficient extracted from $\mathcal{E}_{(3/2, 3/2)}$ is exactly the same as the three-loop contribution to the type IIA theory that was extracted from the one-loop ($L = 1$) supergravity amplitude in [6, 7] (and will be reviewed later in this paper) — which is in agreement with string theory expectations.

2.3 The L -loop four-graviton amplitude compactified on a two-torus

We turn now to consider the expression for the L -loop contribution to four-graviton scattering in asymptotically flat eleven-dimensional space-time compactified on a two-torus of volume \mathcal{V} and complex structure Ω . In the following the ultraviolet divergences will be regulated by simply introducing a momentum cut-off Λ , as we did for the one-loop and

two-loop cases, although we expect that the precise details of how this is implemented will not be relevant to the following general arguments.⁵ To begin with, we will describe the analytic contributions to the amplitude, which translate into local terms, S_L , in the action. The derivative expansion will be written as $S_L = \sum_{v=0}^{\infty} S_L^{(\beta_L+3+v)}$, where v labels the power of the derivatives in the expansion. Dimensional analysis determines that the leading low energy contribution ($v = 0$) has the form

$$S_L^{(\beta_L+3)} = l_P^{9(L-1)} \sum_{w=0}^{w_L} \Lambda^{9L-6-2\beta_L-w} \int d^9x \sqrt{-G^{(9)}} \mathcal{V}^{1-\frac{w}{2}} f_{(\frac{w}{2}, \beta_L)}(\Omega, \bar{\Omega}) D^{2\beta_L} R^4, \quad (2.10)$$

where $-G^{(9)}$ is the determinant of the nine-dimensional metric and l_P is the eleven-dimensional Planck distance. The function $f_{(w/2, \beta_L)}(\Omega, \bar{\Omega})$ is invariant under modular transformations (large diffeomorphisms) of the two-torus and, as noted earlier, direct Feynman diagram calculations give $\beta_1 = 0$ and $\beta_2 = 2$. Furthermore the R^4 interaction is protected by supersymmetry from renormalization beyond one loop (see, for example, [13]) so $\beta_L \geq 2$ for $L > 2$. The integer w determines the power of Λ , with $w = 0$ for the leading divergence and $w > 0$ for subdivergences, which are associated with corresponding inverse powers of \mathcal{V} (possible $(\ln \Lambda)^n$ factors are not explicitly shown since they are not seen in the following power-counting analysis). The values of w that are summed over are bounded by $w_L \leq 9L - 6 - 2\beta_L$ and depend on the nature of these divergences. As described earlier, a power of Λ in (2.10) and subsequent equations will be taken to represent the undetermined renormalised value that results when divergences are canceled by adding appropriate counterterms. This value will be assumed to be a constant in Planck units.

Higher order terms in the derivative expansion are obtained by expanding the amplitude in powers of the Mandelstam invariants S , T and U . Dimensional analysis gives an expression for S_L as an infinite series of powers of $\mathcal{V}D^2$ in which the general term ($v \geq 0$) has the form

$$S_L^{(\beta_L+v+3)} = \sum_{w=0}^{w_L} l_P^{9(L-1)} \Lambda^{9L-6-2\beta_L-w} \int d^9x \sqrt{-G^{(9)}} \mathcal{V}^{1-\frac{w}{2}} f_{(\frac{w}{2}-v, \beta_L+v)}(\Omega, \bar{\Omega}) (\mathcal{V}D^2)^v D^{2\beta_L} R^4, \quad (2.11)$$

We may now define

$$k = v + \beta_L \quad q = \frac{w}{2} - v. \quad (2.12)$$

and write (2.11) as

$$S_L^{(k+3)} = \sum_{q=\beta_L-k}^{q_L} l_P^{9(L-1)} \Lambda^{9L-6-2k-2q} \int d^9x \sqrt{-G^{(9)}} \mathcal{V}^{1-q} f_{(q,k)}(\Omega, \bar{\Omega}) D^{2k} R^4. \quad (2.13)$$

Note that $k \geq \beta_L \geq 0$, while q can be of either sign and has the upper limit $q_L = \beta_L - k + w_L/2$. Equation (2.13) involves a sum of terms with different values of q that

⁵ A simple momentum cut-off obviously breaks the local symmetries, which would then have to be restored by the counterterms — a procedure that would be very difficult to implement explicitly in practise.

depends on details of which values of w arise in the expressions for the L -loop amplitude. Terms with positive powers of q are suppressed in the large- \mathcal{V} limit while those with negative powers diverge as $\mathcal{V} \rightarrow \infty$. This distinctive behaviour will be discussed further below.

The modular function $f_{(q,k)}$ is undetermined by our general analysis. However, duality with string theory requires that it has an expansion in powers of Ω_2^{-2} in order to correspond to the string perturbation expansion in powers of g_B^2 , together with an infinite series of non-perturbative D -instanton terms that has the general form

$$f_{(q,k)}(\Omega, \bar{\Omega}) = c_0^{q,k} \Omega_2^{a_{q,k}} + c_1^{q,k} \Omega_2^{a_{q,k}-2} + \dots + c_h^{q,k} \Omega_2^{a_{q,k}-2h} + \dots + \sum_{k \neq 0, l \geq |k|} b_{k,l}^{q,k} e^{2\pi i k \Omega_1} e^{-2\pi l \Omega_2}. \quad (2.14)$$

The exact values of the coefficients and powers in this expression cannot be determined purely from perturbative supergravity. However, as we will shortly see, the value of the constant $a_{q,k}$ is determined if we assume duality with string theory, so that the leading power of Ω_2 coincides with tree level (genus 0) in IIB string theory, and h is the genus in the string theory interpretation.

2.4 Relation to type II string theories

In order to discuss the string theory interpretation of the L -loop amplitude we will first review the well-known dictionary that expresses the correspondence between eleven-dimensional supergravity and type II string theories [1–3]. The eleven-dimensional metric is related to the IIA string metric by

$$(ds)^2 = G_{MN}^{(11)} dx^M dx^N = \frac{l_P^3}{l_s^2} [R_{11}^{-1} (ds_{\text{IIA}}^{(9)})^2 + R_{11}^{-1} G_{ij} dx^i dx^j], \quad (2.15)$$

where $M, N = 0, 1, \dots, 9, 11$ are eleven-dimensional Lorentz vector indices, $ds_{\text{IIA}}^{(9)}$ is the type IIA element of length in nine dimension ($M, N = 0, 1, \dots, 8$) and $i, j = 9, 11$. The torus metric G_{ij} can be written in the form

$$G = \frac{\mathcal{V}}{\Omega_2} \begin{pmatrix} |\Omega|^2 & -\Omega_1 \\ -\Omega_1 & 1 \end{pmatrix}, \quad (2.16)$$

where the complex structure and volume of the torus are given by $\Omega = \Omega_1 + i\Omega_2$ and $\mathcal{V} = R_9 R_{11}$, where R_9 and R_{11} are the radii of its cycles. In the following we will only consider the special cases in which the indices on the polarization tensors and momenta are in the non-compact nine-dimensional directions, $0, 1, \dots, 8$. From this, and noting the factor of R_{11}^{-1} in front of the nine-dimensional metric on the right-hand side of (2.15), it follows that the eleven-dimensional Mandelstam invariants for four-graviton scattering⁶ are related to those of string theory by

$$l_P^2 S = \frac{R_{11}}{l_P} l_s^2 s, \quad l_P^2 T = \frac{R_{11}}{l_P} l_s^2 t, \quad l_P^2 U = \frac{R_{11}}{l_P} l_s^2 u, \quad (2.17)$$

⁶These invariants are defined by $S = -\eta^{MN} (k_1 + k_2)_M (k_1 + k_2)_N$, $T = -\eta^{MN} (k_1 + k_2)_M (k_1 + k_2)_N$, $U = -\eta^{MN} (k_1 + k_2)_M (k_1 + k_2)_N$, where k_r ($r = 1, 2, 3, 4$) are the moment of the four gravitons and η_{MN} is the eleven-dimensional Minkowski metric.

where lower case letters denote the string theory invariants in the string frame.

The parameters of the corresponding type IIB string theory on a circle of radius r_B are given by

$$r_B^{-1} = R_9 R_{11}^{\frac{1}{2}} l_P^{-\frac{3}{2}}, \quad C^{(0)} = \Omega_1, \quad e^{-\phi_B} = \Omega_2 = \frac{R_9}{R_{11}}, \quad (2.18)$$

where $C^{(0)}$ is the Ramond-Ramond zero-form and e^{ϕ_B} is the IIB coupling, and r_B is the dimensionless length (in string units) of the radius of compactification from ten to nine dimensions. Note that the torus volume is given by

$$\mathcal{V} = R_9 R_{11} = e^{\phi_B/3} r_B^{-\frac{4}{3}} l_P^2. \quad (2.19)$$

The type IIA theory is obtained using the identifications

$$r_A = R_9 R_{11}^{\frac{1}{2}} l_P^{-\frac{3}{2}}, \quad C^{(9)} = \Omega_1, \quad e^{\phi_A} = R_{11}^{\frac{3}{2}} l_P^{-\frac{3}{2}}, \quad (2.20)$$

where $C^{(9)}$ is the component of the Ramond-Ramond one-form along the compact dimension of radius R_9 , r_A is the dimensionless length (in string units) of the radius of compactification from ten to nine dimensions. It follows that the IIB parameters are related to those of IIA by

$$r_B = r_A^{-1}, \quad r_B e^{-\phi_B} = e^{-\phi_A}. \quad (2.21)$$

Equation (2.13) can easily be rewritten in terms of IIB string theory coordinates in the string frame as

$$S_L^{(k+3)B} = \sum_{q=\beta_L-k}^{q_L} S_{m,q}^{(k+3)B}, \quad (2.22)$$

where

$$S_{m,q}^{(k+3)B} = l_s^{2k-1} \hat{\Lambda}^m \int d^9 x \sqrt{-g_B^{(9)}} r_B^{-1+\frac{4q}{3}-\frac{2k}{3}} e^{(\frac{2k}{3}-\frac{q}{3})\phi_B} f_{(q,k)}(\Omega, \bar{\Omega}) D^{2k} R^4, \quad (2.23)$$

and the complex structure, Ω , is now interpreted as the complex IIB coupling constant. In this equation we have introduced the dimensionless cut-off $\hat{\Lambda} = l_P \Lambda$ expressed in M-theory units and the power of the cut-off is denoted by

$$m = 9L - 6 - 2k - 2q. \quad (2.24)$$

The factor of $e^{(2k/3-q/3)\phi_B}$ is absent in the Einstein frame. The perturbative string theory contributions are obtained by substituting (2.14) into (2.23). In order for the leading power to correspond to string tree level behaviour, $\Omega_2^2 = e^{-2\phi_B}$, we need to set

$$a_{q,k} = 2 + \frac{2k}{3} - \frac{q}{3}. \quad (2.25)$$

Terms proportional to r_B , which give finite contributions in the ten-dimensional limit, $r_B \rightarrow \infty$, are obtained by setting

$$\frac{4q}{3} - \frac{2k}{3} = 2, \quad (2.26)$$

which fixes the value of q for a given k . Thus a non-zero L -loop eleven-dimensional supergravity contribution to $D^{2k}R^4$ in the IIB ten-dimensional limit comes from diagrams which have a $\Lambda^{9(L-1)-3k}$ dependence on the cut-off (for $L > 1$, since $9(L-1) - 3k$ is negative when $L = 1$).

Terms with $4q/3 - 2k/3 < 2$ vanish in the large- r_B limit whereas those with $4q/3 - 2k/3 > 2$ diverge in the large- r_B limit. The terms that grow with powers of r_B have to resum in a manner that generates the non-analytic logarithmic thresholds in ten dimension as discussed for the case of the leading supergravity threshold in [7], as we will see in section 4.2. For terms that are proportional to r_B^{1+p} the relation (2.26) generalizes to

$$\frac{4q}{3} - \frac{2k}{3} = 2 + 2p. \quad (2.27)$$

The expression (2.13) can also be rewritten in terms of IIA string-frame coordinates as

$$S_L^{(k+3)A} = \sum_{q=\beta_L-k}^{q_L} S_{m,q}^{(k+3)A}, \quad (2.28)$$

where

$$S_{m,q}^{(k+3)A} = l_s^{2k-1} \hat{\Lambda}^m \int d^9x \sqrt{-g_A^{(9)}} r_A^{1-q+a_{q,k}} e^{(\frac{2k}{3}-\frac{q}{3})\phi_A} (c_0 e^{-a_{q,k}\phi_A} + c_1 r_A^{-2} e^{-(a_{q,k}-2)\phi_A} + \dots + c_h r_A^{-2h} e^{-(a_{q,k}-2h)\phi_A} + \dots) D^{2k}R^4. \quad (2.29)$$

In this expression we have exhibited the perturbative terms in the expansion of $f_{(q,k)}(\Omega, \bar{\Omega})$ (2.14) (for clarity, we have set $c_i^{q,k} \equiv c_i$, $i = 0, \dots, h$). Once again, if we assume that the leading term is the tree-level string theory interaction we have $a_{q,k} = 2 + 2k/3 - q/3$, so that

$$S_{m,q}^{(k+3)A} = l_s^{2k-1} \hat{\Lambda}^m \int d^9x \sqrt{-g_A^{(9)}} r_A^{3-\frac{4q}{3}+\frac{2k}{3}} (c_0 e^{-2\phi_A} + c_1 r_A^{-2} + \dots + c_h r_A^{-2h} e^{2(h-1)\phi_A} + \dots) D^{2k}R^4, \quad (2.30)$$

In this case the condition on q that a given term should have a finite large- r_A limit requires

$$\frac{4q}{3} - \frac{2k}{3} = 2 - 2h. \quad (2.31)$$

The value of q satisfying this equation depends on the genus, in contrast to the IIB case (2.26). Furthermore, in the limit $r_A \rightarrow \infty$, the D -instanton terms in (2.14) vanish (since the D -instanton action is proportional to r_A). More generally, a term behaving as r_A^{1+p} is selected by choosing

$$\frac{4q}{3} - \frac{2k}{3} = 2 - 2h - p \quad (2.32)$$

3. Higher derivative interactions in type IIB

In the absence of further information the modular functions $f_{(q,k)}$ are undetermined and can contain arbitrary powers of the string coupling, Ω_2^{-1} . Therefore, the general structure of the IIB action in (2.23) does not lead to non-renormalisation statements (of the kind that we will find later in the IIA case) but a number of interesting systematic statements can be made. We will first discuss terms that grow linearly with r_B and contribute a finite quantity to the ten-dimensional string effective action. We will then consider terms involving higher powers of r_B , which diverge in ten dimensions but can be resummed to give finite, nonlocal contributions to the ten-dimensional theory that correspond to threshold cuts in the amplitude.

3.1 Terms that are finite as $r_B \rightarrow \infty$

The terms proportional to r_B that are finite in the large- r_B limit have (from (2.26)) $q = 3/2 + k/2$, which means that the dilaton prefactor in (2.23) is $e^{(2k-q)\phi_B/3} = e^{(k-1)\phi_B/2}$. This means that the modular functions, $f_{(q,k)}(\Omega, \bar{\Omega})$, that contribute to tree-level string scattering have an expansion that starts with $\Omega_2^{(3+k)\phi_B/2}$ (since $a_{q,k} = (k+3)/2$). More information is needed to determine the particular modular functions that arise for any value of k . For the cases $k = 0, 2, 3$ the modular functions, determined by explicitly compactifying one and two-loop eleven-dimensional supergravity on a torus, are known to be $f_{(3/2,0)} = Z_{3/2}$, $f_{(5/2,2)} = Z_{5/2}$, $f_{(3,3)} = \mathcal{E}_{(3/2,3/2)}$, where the generalized Eisenstein series satisfy the Laplace or Poisson equations described in the introduction. There is impressive agreement between the coefficients of the perturbative terms contained in these functions and those that have been calculated at tree-level and one-loop in string theory. There is also internal consistency between the IIA and IIB calculations. Whereas the R^4 term is protected from getting any contribution from $L > 1$ supergravity loops, there is no known reason for the $L = 2$ calculations of the $D^4 R^4$ and $D^6 R^4$ interactions to be protected. However, the agreement of these coefficients suggests that there is some as yet undiscovered non-renormalisation condition.

3.2 Terms that diverge as $r_B \rightarrow \infty$

Apart from terms that are finite or vanish in the large- r_B limit, (2.23) also allows for terms that diverge in the ten-dimensional limit. Such terms that are essential for building up the non-analytic threshold behaviour of the amplitude in ten dimensions [7]. The form of these non-analytic terms is highly constrained by unitarity but this is obviously very difficult to analyze explicitly in the general case. However, if we restrict our considerations to two-particle thresholds we will be able to pinpoint some essential features of certain infinite series' of terms. The s -channel two-particle discontinuity of the amplitude in ten dimensions is given by the integral over two-particle phase space of the product of two four-graviton amplitudes (we are not here concerned with exact coefficients),

$$\text{Disc}_s A_4(s, t, u) \sim \int d^{10}q A_4(k_1, k_2, q, -q - k_1 - k_2) A_4^\dagger(k_3, k_4, -q, q + k_3 + k_4) \\ \delta^{(10)}(q^2) \delta^{(10)}((q + k_1 + k_2)^2) \theta(q_0) \theta((q + k_1 + k_2)_0). \quad (3.1)$$

In nine dimensions the integral includes the sum over intermediate states that include an infinite sequence of massive Kaluza-Klein two-particle states.

The low energy expansion of (3.1) involves expanding each factor of A_4 in powers of α' , so the derivative expansion of the amplitude feeds back into the expression for the normal thresholds. We will distinguish the analytic part of the amplitude, A^{an} , from the nonanalytic part, A^{nonan} , that has singularities, which is associated with thresholds of various kinds. So we will write

$$A_4 = A_4^{\text{an}} + A_4^{\text{nonan}}. \quad (3.2)$$

The A_4^{an} term encodes local terms in the effective action whereas A_4^{nonan} contains normal thresholds that correspond to non-local terms. For the type IIB theory in ten dimensions the expansion of A_4^{an} has the schematic form (that was reviewed in the introduction)⁷

$$A_4^{\text{an}} \sim \alpha'^{-4} \Omega_2^2 A_4^{\text{Born}} + \left(\alpha'^{-1} \Omega_2^{\frac{1}{2}} Z_{\frac{3}{2}} + \alpha' \Omega_2^{-\frac{1}{2}} Z_{\frac{5}{2}} s^2 + \alpha'^2 \Omega_2^{-1} \mathcal{E}_{(\frac{3}{2}, \frac{3}{2})} s^3 + O(\alpha'^3) \right) R^4, \quad (3.3)$$

where A_4^{Born} is the tree-amplitude that corresponds to the Einstein-Hilbert part of the action. Substituting this into (3.1) leads to expressions for the first few threshold contributions to the amplitude. For the purposes of this section we need only describe a few of the many threshold terms that arise up to order $(\alpha')^7$,

$$A_4^{\text{nonan}} \sim \left(s \ln(-\alpha' s) + \alpha'^3 \Omega_2^{-\frac{3}{2}} Z_{\frac{3}{2}} s^4 \ln(-\alpha' s) + \alpha'^4 \Omega_2^{-2} s^5 \ln^2(-\alpha' s) \right. \\ \left. + \alpha'^5 \Omega_2^{-\frac{5}{2}} Z_{\frac{5}{2}} s^6 \ln(-\alpha' s) + \alpha'^6 \Omega_2^{-3} (Z_{\frac{3}{2}} \times Z_{\frac{3}{2}} + \mathcal{E}_{(\frac{3}{2}, \frac{3}{2})}) s^7 \ln(-\alpha' s) + \dots \right) R^4. \quad (3.4)$$

Note that this expression includes a $s^5 \ln^2(-\alpha' s) R^4$ that is the first contribution that arises by substituting a loop contribution into one of the factors of A_4 inside the integral on the right-hand side of (3.1). This term has a double discontinuity and arises from a two-loop supergravity diagram.

The separation between analytic and nonanalytic terms in (3.2) is ambiguous since unitarity does not determine the scale of the argument of the $\ln(-s)$'s in (3.4). We have chosen to normalize the arguments of each logarithm so that it vanishes at the string scale, $s = 1/\alpha'$. Changing that scale amounts to a shift in the value of the coefficient of a corresponding analytic term. For example, $s^4 \ln(-\alpha' s/A) = s^4 \ln(-\alpha' s) - s^4 \ln A$. In terms of the effective action this scale defines the scale for the Wilsonian cut-off, which is an arbitrary parameter in the action.

It is, of course, the complete amplitude that is supposed to be invariant under duality transformations, $\text{SL}(2, Z)$ for type IIB in ten dimensions (or $\text{SL}(2, Z) \times R$, in nine dimensions). This is manifest in Einstein frame whereas the preceding expressions were written in string frame. The transformation to Einstein frame requires the replacement of

⁷Constant coefficients have been omitted and dependence on t and u has been suppressed in this and the following equations.

the ten-dimensional string metric, $g_{\mu\nu}$ by $e^{\phi_B/2} g_{\mu\nu} = \Omega_2^{-1/2} g_{\mu\nu}$, which implies that s is replaced by $\Omega_2^{1/2} s$. After taking into account the fact that A_4 contains a density factor $\sqrt{-g}$ we find that (3.3) and (3.4) become, in Einstein frame,

$$A_4^{Ean} \sim \alpha'^{-4} A_4^{\text{Born}} + \left(\alpha'^{-1} Z_{\frac{3}{2}} + \alpha' Z_{\frac{5}{2}} s^2 + \alpha'^2 \mathcal{E}_{(\frac{3}{2}, \frac{3}{2})} s^3 + \alpha'^3 Z_{\frac{3}{2}} s^4 \ln(\Omega_2^{\frac{1}{2}}) + \alpha'^4 s^5 \ln^2(\Omega_2^{\frac{1}{2}}) + \dots \right) R^4. \quad (3.5)$$

and

$$A_4^{Enonan} \sim \left(s \ln(-\alpha' s) + \alpha'^3 Z_{\frac{3}{2}} s^4 \ln(-\alpha' s) + \alpha'^4 s^5 \ln^2(-\alpha' s) + \alpha'^4 s^5 \ln(-\alpha' s) \ln(\Omega_2^{\frac{1}{2}}) + \alpha'^5 Z_{\frac{5}{2}} s^6 \ln(-\alpha' s) + \alpha'^6 (Z_{\frac{3}{2}} \times Z_{\frac{3}{2}} + \mathcal{E}_{(\frac{3}{2}, \frac{3}{2})}) s^7 \ln(-\alpha' s) + \dots \right) R^4. \quad (3.6)$$

All the factors involving explicit Ω_2 's have disappeared, but additional terms with factors of $\ln \Omega_2$ are now present. These contribute to the $\ln \Omega_2$ terms that are analytic in the Mandelstam invariants in A_4^{Ean} . The above equations are schematic and omit the dependence on t and u . Inserting this, and using $s + t + u = 0$, accounts for the absence of the $s \ln \Omega_2$ term in (3.5).

Certain terms in (3.5) and (3.6) are manifestly $\text{SL}(2, Z)$ invariant, but those containing factors of $\ln \Omega_2$ do not transform properly since $\ln \Omega_2 \rightarrow \ln(\Omega_2/|c\Omega + d|^2)$, under the $\text{SL}(2, Z)$ transformation, $\Omega \rightarrow (a\Omega + b)/(c\Omega + d)$, ($a, b, c, d \in Z$ with $ad - bc = 1$). The lowest order term of this kind is $\alpha'^3 Z_{3/2} s^4 \ln(\Omega_2^{1/2})$, which translates into a $D^8 R^4$ interaction. Modular invariance of the amplitude obviously requires this to be part of a modular function, which remains to be determined. We expect that this modular function should be the solution of a Poisson equation on moduli space, generalizing the structure of the $D^6 R^4$ interaction (as will be discussed further in [17]).

3.2.1 Terms that sum to the ten-dimensional $s \ln(-\alpha' s)$ threshold

The leading two-particle threshold, which arises at genus $h = 1$ is just the massless supergravity normal threshold, which has the form $(-\alpha' s)^{1/2}$ in nine dimensions and $s \ln(-\alpha' s)$ in ten dimensions. The change in analytic behaviour is due to a condensate of the massive Kaluza-Klein two-particle thresholds that can be symbolically represented by $\sum_n c_n (-s + 4n^2/r_B^2)^{1/2}$. For finite r_B this has a derivative expansion expressed as the sum of terms of the form $\sum_{k=2}^{\infty} d_k r_B^{-1+2k} D^{2k} R^4$ whereas as $r_B \rightarrow \infty$ the sum over n generates the $s \ln(-\alpha' s)$ behaviour of the ten-dimensional theory. The analogous infinite series of dilaton-independent powers of $s r_A^2$ is explicit in the second line of equation (3.29) in [7], and will be reviewed in subsection 4.2.1 as well as in appendix B. In the IIB case these terms arise from supergravity loops with different values of L (whereas, as we will see, in the IIA case the whole series is contained purely in the one-loop ($L = 1$) supergravity amplitude). The power of r_B^{-1+2k} is obtained by setting $q = 2k$ in (2.23). Furthermore,

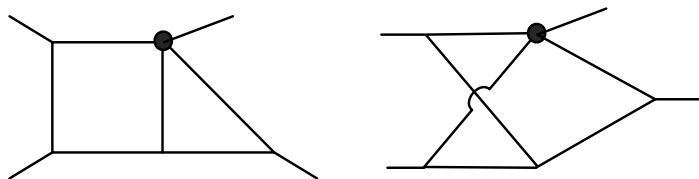


Figure 3: Two diagrams with one-loop subdivergences that contain terms that contribute to the $s \ln(-\alpha' s)$ threshold in the ten-dimensional IIB limit.

taking $f_{(q,k)}$ to be constant ensures the genus one condition. The expression for these terms in eleven-dimensional coordinates is proportional to

$$\int d^9 x \sqrt{-G^{(9)}} \mathcal{V}^{1-2k} D^{2k} R^4, \quad (3.7)$$

where $k > 1$. The supergravity diagrams that contribute to these terms are those for which

$$3L = 2k + 2 + \frac{m}{3}. \quad (3.8)$$

Since L is an integer, equation (3.8) only has solutions for values of m that are multiples of 3. We see that the terms in this series have $L > 2k/3$. The lowest value arises at two loops ($L = 2$) for $k = 2$ and $m = 0$. The precise value of this finite two-loop diagram was evaluated in [7] and gave exactly the same result as the $k = 2$ term in the IIA theory expression (4.8), with r_A replaced by r_B . This agreement suggests that the higher-loop contributions, which are obtained by increasing m by multiples of 9, do not contribute to the $D^4 R^4$ interaction, thereby explaining why the two-loop result is exact. The $D^6 R^4$ interaction ($k = 3$) arises from $L = 3$ with $m = 3$, which corresponds to a one-loop subdivergence of a three-loop diagram as in figure 3.

In the IIA case we will see that all the terms that resum to the threshold cut arise from the one loop ($L = 1$) supergravity amplitude.

3.2.2 Terms that sum to the ten-dimensional $\alpha'^3 s^4 \ln(-\alpha' s)$ threshold at genus one and two

We can also determine in surprising detail properties of the infinite series of contributions that sums up to give the $s^4 \ln(-\alpha' s)$ threshold required by unitarity in the ten-dimensional IIB theory at genus one and two ($h = 1$ and $h = 2$). These must arise by summing the Kaluza-Klein thresholds in nine dimensions of the form $(-s + 4n^2/r_B^2)^{7/2}$. The low energy expansion is now a series of terms that have the form $r_B^{2k-7} D^{2k} R^4$ in (2.23) (again this will be seen explicitly in the IIA case in section 4.2. The r_B -dependence requires $q = 2k - 9/2$, which results in a dilaton prefactor of $e^{3\phi_B/2}$, which is independent of k . This means that the modular functions in these terms in (2.23) must have expansion of the form

$$f_{(2k-\frac{9}{2},k)}(\Omega, \bar{\Omega}) = c_0 \Omega_2^{\frac{3}{2}} + c_1 \Omega_2^{-\frac{1}{2}} + \text{non-pert}. \quad (3.9)$$

There can be no higher powers of Ω_2^{-2} since these would generate $s^4 \ln(-\alpha' s)$ threshold behaviour at higher genus, which is not permitted by unitarity. So we conclude that for all q , $f_{(q,k)}$ must be a $\text{SL}(2, Z)$ scalar function that has an expansion of the form (3.9). But this is completely consistent with the identification

$$f_{(q,k)}(\Omega, \bar{\Omega}) = c_{q,k} Z_{\frac{3}{2}}(\Omega, \bar{\Omega}), \quad (3.10)$$

for some constant $c_{q,k}$ that we can fix by requiring the sum of the infinite series to reproduce the $O(\alpha'^3)$ term in the expansion of the ten-dimensional unitarity expression (3.1). When expressed in supergravity coordinates, the terms that contribute to this threshold have the form

$$\int d^9 x \sqrt{-G^{(9)}} \mathcal{V}^{1-2k+\frac{9}{2}} Z_{\frac{3}{2}}(\Omega, \bar{\Omega}) D^{2k} R^4. \quad (3.11)$$

In this case the number of string loops is related to k by

$$3L = 2k - 4 + \frac{m}{3}. \quad (3.12)$$

For example, there is a finite ($m = 0$) contribution from two loops ($L = 2$) to $D^{10} R^4$, which has $q = 2k - 9/2 = 11/2$.

In order to obtain the $s^5 \ln^2(-\alpha' s)$ threshold at genus-two ($h = 2$) required by unitarity we need to sum a series of terms with powers r_B^{2k-9} , which requires $q = 2k - 6$. This leads to a dilaton factor of $e^{2\phi_B}$ in (2.23). Therefore, if $f_{(q,k)}$ are constants in this case, for all values of q and k the sum will contribute only at genus two, as required. However, we have not studied the set of coefficients that generate a $\ln^2(-\alpha' s)$ factor.

In order to obtain the $\alpha'^5 s^6 \ln(-\alpha' s)$ threshold required by unitarity we need to sum a series with powers $r_B^{2k-11} D^{2k} R^4$ in (2.23). The r_B -dependence requires $q = 2k - 15/2$, which results in a dilaton prefactor of $e^{5\phi_B/2}$, which is independent of k . This means that the modular functions in these terms in (2.23) must have expansion of the form

$$f_{(2k-\frac{15}{2},k)}(\Omega, \bar{\Omega}) = c_0 \Omega_2^{\frac{5}{2}} + c_1 \Omega_2^{-\frac{3}{2}} + \text{non-pert}. \quad (3.13)$$

There can be no higher powers of Ω_2^{-2} since these would generate $s^6 \ln(-\alpha' s)$ threshold behaviour at higher genus, which is not permitted by unitarity (as can easily be seen by substituting (3.3) into (3.1)). So we conclude that for all q , $f_{(2k-15/2,k)}$ must be a $\text{SL}(2, Z)$ scalar function that has an expansion of the form (3.13). But this is completely consistent with the identification

$$f_{(2k-\frac{15}{2},k)}(\Omega, \bar{\Omega}) = c Z_{\frac{5}{2}}(\Omega, \bar{\Omega}), \quad (3.14)$$

for some constant c that can be fixed by requiring the sum of the infinite series to reproduce the $O(\alpha'^3)$ term in the expansion of the ten-dimensional unitarity expression (3.1). At higher orders in α' than those considered up to this point the analysis rapidly gets very complicated.

4. Higher derivative interactions in type IIA

Whereas the IIB coupling is the dimensionless ratio of the radii of the torus, the IIA coupling is determined by the single length scale R_{11} . This will lead to powerful restrictions on the possible powers of the coupling in the IIA action (2.30).

4.1 Terms that are finite as $r_A \rightarrow \infty$ — non-renormalisation conditions

To be specific we will first specialize to those terms that are proportional to r_A and so have a finite ten-dimensional ($r_A \rightarrow \infty$) limit. Therefore q will be taken to satisfy (2.31),

$$2q = 3 + k - 3h, \quad (4.1)$$

or, from the definitions (2.12),

$$w = 3 + 3k - 3h - 2\beta_L. \quad (4.2)$$

Since $w \geq 0$ we see that

$$h \leq k + 1 - \frac{2\beta_L}{3}. \quad (4.3)$$

The ten-dimensional limit for a term with string genus h has the form

$$S_{m,h}^{(k+3)A} = l_s^{2k-2} \hat{\Lambda}^m \int d^{10}x \sqrt{-g_A} c_h e^{2(h-1)\phi_A} D^{2k} R^4 \quad (4.4)$$

(recall that $m = 9L - 6 - 2k - 2q = 3(3L - 3 - k + h)$). Since $\beta_1 = 0$ while $\beta_L \geq 2$ for $L \geq 2$ we need to distinguish the case $L = 1$ from $L > 1$.

(a) When $L = 1$ we need to choose $\beta_L = 0$ in (4.2) so that $h \leq k + 1$. If $h = k + 1$ it follows that $w = 0$ and so $m = 3$ and (2.10) contains a factor of Λ^3 , corresponding to the one-loop divergence. However, the Λ^3 divergence of the one-loop amplitude [6, 7] is independent of the momenta and only contributes to the $k = 0, h = 1$ term. The $h = k$ terms have $w = 3$ and are finite ($m = 0$ so (2.10) is independent of Λ). The contributions of these terms, given in [6, 7], is

$$2\zeta(3)R^4 + 8\pi^2 \sum_{h=2}^{\infty} l_s^{2h-2} \int d^9x \sqrt{-g_A^{(9)}} r_A \Gamma(h-1) \zeta(2h-2) \frac{e^{2(h-1)\phi_A}}{h!} D^{2h} R^4. \quad (4.5)$$

(b) For $L > 1$ we are assuming $\beta_L \geq 2$ and so it follows from (4.3) that

$$h < k - \frac{1}{3}. \quad (4.6)$$

Clearly, these ten-dimensional IIA results could have been obtained more directly by considering compactification from eleven to ten dimensions on a circle of radius R_{11} , instead of considering a torus. In that case the starting expression (2.11) is replaced by

$$S_L^{(\beta_L+v+3)} = l_P^{9(L-1)} \sum_{w=0}^{w_L} \Lambda^{9L-6-2\beta_L-w} \int d^{10}x \sqrt{-G^{(10)}} R_{11}^{1-w} (R_{11}^2 D^2)^v D^{2\beta_L} R^4. \quad (4.7)$$

Translating into IIA string variables immediately leads to (4.4) with the same set of equations (4.1) – (4.3) (where $k = v + \beta_L$ and $q = w/2 - v$, as given in (2.12)).

We therefore conclude that type IIA string theory in ten dimensions satisfies the following strong conditions (for $k \geq 1$):

- *There are no contributions with string loop genus $h > k$.*

- *The contributions with $h = k$ are determined exactly by finite contributions from the derivative expansion of the one-loop ($L = 1$) diagram in eleven dimensions.*
- *Contributions with $h < k$ are permitted and may arise from any number of supergravity loops greater than one ($L > 1$).*

It is of interest to consider what would happen if extra powers of $D^2 \sim S, T, U$ were to factor out of the L -loop amplitudes of eleven-dimensional supergravity for $L > 2$ (in other words, if $\beta_L > 2$ for $L > 2$). For example, if it turned out that every extra loop had an extra power of D^2 then we would have $\beta_L = L$. In that case (4.2) would imply $w = 3 + 3k - 3h - 2L$, so that $h \leq k + 1 - 2L/3$. This would mean that the bound still requires $h \leq k$. Furthermore, since $h \geq 0$, all perturbative contributions to $D^{2k}R^4$ would be obtained from $L \leq 3(k+1)/2$.

Although we have obtained an upper bound on h , we have not shown that it also satisfies the obvious lower bound, $h \geq 0$, that ensures there are no terms more singular than the genus-zero tree-level terms. Nevertheless, there are indications from the explicit $L = 1$ and $L = 2$ calculations that contributions with $h < 0$ are, in fact, absent. Note also that the type IIA bound does not explain why certain terms are absent in string perturbation theory, such as D^4R^4 at one loop ($k = 2, h = 1$). In the IIB case, where the perturbative contributions to D^4R^4 are contained in $Z_{5/2}$, the $h = 1$ component is explicitly absent.

Since the considerations up to this point have mostly been purely dimensional one might expect similar statements to hold for other terms of the same dimension. For example, this ties in with the argument in [6] that powers of the curvature of the form R^{3n+1} are the only ones that have a nonzero eleven-dimensional limit. Although the case $k = 1$ is absent for D^2R^4 since $s + t + u = 0$, a term of the same dimension might be present and if the above arguments extend to this case, it would have a contribution of maximum possible genus $h = 1$.

One consequence of our statements is that the low energy behaviour of the genus- h loop contribution to the type IIA four-graviton effective amplitude has the form $s^h R^4(1 + O(s))$ (when $h > 1$). This is a very powerful condition, which indicates that the low-energy ten-dimensional theory is much less divergent than it might have been. This will be discussed in more detail in section 5.

4.2 Terms that diverge as $r_A \rightarrow \infty$

A general nine-dimensional contribution behaves as r_A^{1+p} , which implies the relation (2.31) and leads to a generalization of the bound (4.6) of the form, $h \leq k - 1/3 - p/2$. Terms that diverge in the $r_A \rightarrow \infty$ limit have $p > 0$ so the bound is stronger. Just as for the type IIB theory, these are the terms that arise from the low energy expansion of the normal thresholds due to massive Kaluza-Klein intermediate states. These terms must sum up to give, in the $r_A \rightarrow \infty$ limit, the appropriate ten-dimensional threshold behaviour. The unitarity relation for the ten-dimensional IIA case has identical perturbative structure to the IIB case (at least to the order considered explicitly in the IIB case) so it is given by (3.1) with the omission of the non-perturbative D -instanton contributions.

4.2.1 Terms that sum to the ten-dimensional $s \ln(-\alpha' s)$ threshold

Just as in the IIB case, the IIA theory must generate a non-analytic $s \ln(-\alpha' s)$ genus-zero massless supergravity threshold in the four-graviton amplitude in ten dimensions. Once again, the leading massless supergravity normal threshold has the form $(-\alpha' s)^{1/2}$ in nine dimensions and $s \ln(-\alpha' s)$ in ten dimensions. The infinite series of dilaton-independent powers of r_A^{-1+2k} is explicit in the second line of equation (3.29) of [7] and has the form

$$r_A^{-1}(-\alpha' s)^{\frac{1}{2}} R^4 - \pi^{-\frac{1}{2}} \sum_{k=2}^{\infty} \frac{\Gamma(k - \frac{1}{2})}{k!} \zeta(2k-1) r_A^{2(k-1)} (-\alpha' s)^k R^4. \quad (4.8)$$

These contributions arise from the expansion of the one-loop amplitude, and resum to the $s \ln(-\alpha' s)$ threshold contribution in the limit $r_A \rightarrow \infty$ (as reviewed in the appendix B). In order to reproduce this in (2.29) we need

$$r_A^{1 - \frac{4q}{3} + \frac{2k}{3}} = r_A^{2k-1}, \quad (4.9)$$

or $2(k+q) = 3$, which is the same condition as for the series of $h = k$ terms leading to (4.5). The two sets of terms (4.5) and (4.8) are very similar when expressed in eleven-dimensional coordinates — they differ only by a ‘9-11’ flip that interchanges R_9 and R_{11} , thereby interchanging the roles of r_A and e^{ϕ_A} .

4.2.2 Terms that sum to the ten-dimensional $\alpha'^3 s^4 \ln(-\alpha' s)$ threshold at genus one and two

As in the IIB case, we here want to sum a series of terms of the form $r_A^{2k-7} D^{2k} R^4$. There should be a contribution with $h = 1$ and one with $h = 2$. When $h = 1$ we see from (2.30) that we need to set $1 - 4q/3 + 2k/3 = -1 + 2k - 6$ in order to get the correct power of r_A . This means that $q + k = 6$, so that there is contribution from two supergravity loops ($L = 2$) with $m = 0$, since $m = 9L - 6 - 2(k+q)$. The $h = 2$ term is obtained by setting $q + k = 9/2$ in order to get the correct power of r_A . In this case the value of L can only be an integer if $m = 3 \bmod 9$. Setting $m = 3$ again results in $L = 2$, but now the contribution comes from a Λ^3 subdivergence, which agrees with the fact that the one-loop subdivergences of the two-loop diagrams generate the correct series of terms to reproduce the ten-dimensional $h = 2$ threshold.

Figure 4 depicts the origin of such threshold terms from two-loop supergravity Feynman diagrams. The string genus-one ($h = 1$) $s^4 \ln(-\alpha' s)$ threshold behaviour comes from resumming the Kaluza-Klein states in one loop while summing over non-zero windings in the second (this picks out the finite part of the second loop proportional to R_{11}^{-3}). The string genus-two ($h = 2$) part of the $s^4 \ln(-\alpha' s)$ threshold is obtained by taking the zero winding number sector (proportional to the Λ^3 subdivergence) in the second loop, and also including the diagram in which a counterterm replaces that loop. The contributions in the figure are therefore proportional to each other, with coefficients that are determined by the coefficients of the tree-level and one-loop terms in the R^4 interaction that are contained in $Z_{3/2}$.

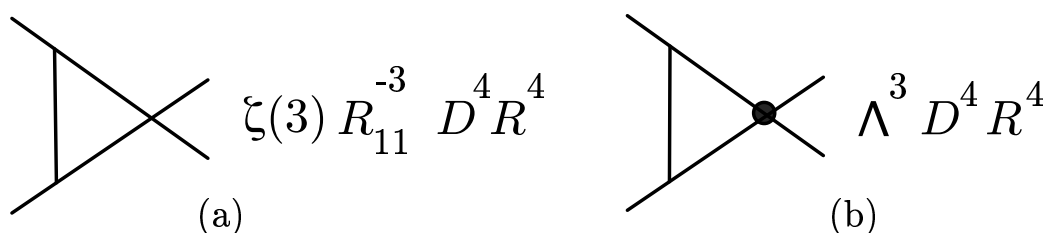


Figure 4: The string one-loop $s^4 \ln(s)$ originates from the two-loop supergravity amplitude by resumming the Kaluza-Klein modes in one loop and using the finite $\zeta(3) D^4 R^4 / R_{11}^3$ contribution from the one-loop amplitude, as in figure (a). The string two-loop contribution arises by picking the Λ^3 counterterm at one-loop, as in figure (b).

Both the diagrams on the right side of figure 4 are proportional to

$$[S^2 I_{\text{Triang}}(S) + T^2 I_{\text{Triang}}(T) + U^2 I_{\text{Triang}}(U)] R^4, \quad (4.10)$$

where $I_{\text{Triang}}(S)$ is a scalar field theory triangle diagram. In appendix B we demonstrate that in ten dimensions this triangle diagram has the non-analytic structure, $I_{\text{Triang}}(s) \sim s^2 \ln(-\alpha' s)$. When multiplied by the appropriate $h = 1$ factor, $\zeta(3) e^{-2\phi_A} s^2 R^4$ (from the finite part of the $L = 1$ one-loop amplitude), or the $h = 2$ factor, $\zeta(2) s^2 R^4$ (from the regulated divergence of the $L = 1$ one-loop amplitude), this gives the expected string threshold contributions. The $h = 1$ piece reproduces the $s^4 \ln(-\alpha' s)$ term found by explicitly analyzing the one-loop string amplitude in appendix B of [11].

On the other hand in nine dimensions with finite r_A we see from (B.9) that the triangle diagram can be expanded in the expected infinite series of positive powers of r_A that again sum up to the nine-dimensional thresholds for Kaluza-Klein states,

$$(-\alpha' s)^{\frac{3}{2}} - \sum_{k=2}^{\infty} \frac{\Gamma(k - \frac{3}{2})}{\Gamma(-\frac{3}{2}) 2^{2k-3} k!} \zeta(2k-3) r_A^{2k-3} (\alpha' s)^k = \sum_{n \in \mathbb{Z}} \left(\frac{4n^2}{r_A^2} - \alpha' s \right)^{\frac{3}{2}}. \quad (4.11)$$

Multiplying this contribution by $\zeta(3) s^2 R^4$ gives the $h = 1$ threshold term,

$$\lim_{r_A \rightarrow \infty} \zeta(3) r_A^{-1} \sum_{n \in \mathbb{Z}} \left(\frac{4n^2}{r_A^2} - \alpha' s \right)^{\frac{3}{2}} s^2 R^4 \sim 2^{-5} \zeta(3) (\alpha' s)^4 \ln(-\alpha' s) R^4 \quad (4.12)$$

(where we have only kept the $\ln(-\alpha' s)$ term on the right-hand side). Similarly, the $h = 2$ part is simply reproduced by multiplying by the regulated $L = 1$ one-loop divergence $\hat{\Lambda}^3 s^2 R^4 \sim \zeta(2) s^2 R^4$.

4.3 Lifting to eleven dimensions

Now that we have seen how the string theory expansion is constrained by duality with eleven-dimensional supergravity we can ask how self-consistency restricts the eleven-dimensional theory — thereby completing the circle. In other words, which higher derivative interactions survive decompactification to eleven dimensions? This is answered by

reexpressing the ten-dimensional IIA terms in (4.4) in terms of eleven-dimensional supergravity on a circle as

$$S_h^{(k+3)} = l_P^{9L-6-3h+k} \Lambda^{9L-6-2k} \int d^{10}x R_{11} \sqrt{-G^{(9)}} R_{11}^{3h-3-k} D^{2k} R^4. \quad (4.13)$$

The terms that have finite non-zero large- R_{11} limits are ones for which

$$k = 3h - 3. \quad (4.14)$$

All other terms either vanish or diverge as $R_{11} \rightarrow \infty$. As we saw when passing from nine to ten dimensions (and as discussed in [7]) the sum of divergent terms must generate the non-analytic thresholds in eleven dimensions that contribute to nonlocal parts of the action.

We now list some examples of terms satisfying (4.14) that contribute to local terms in the eleven-dimensional action and how they emerge from the supergravity calculations.

- The R^4 interaction ($k = 0$) only gets a contribution from one loop ($L = 1$) eleven-dimensional supergravity (since all $L > 1$ have $k > 0$). This is the genus-one ($h = 1$) term that has $m = 3$ and so is proportional to Λ^3 . The precise value of the counterterm that cancels this divergence was determined in [4].
- The first non-zero value of k satisfying (4.14) is $k = 3$, which contributes to the genus-two ($h = 2$) IIA string action. From (4.4) we see that this gets a contribution from two-loop supergravity ($L = 2$) with $m = 9L - 6 - 2k = 6$, so it is proportional to Λ^6 . The precise value of the counterterm needed to subtract this dependence on the cut-off was determined in [5], where the corresponding $h = 2$ coefficient of the $D^6 R^4$ interaction was determined in the IIB theory. The known equality of the IIA and IIB four-graviton string theory amplitudes at two loops therefore fixes the coefficient in the IIA theory, which then lifts to eleven dimensions. As mentioned earlier, the precise matching of the other $D^6 R^4$ coefficients in the IIB theory encourages us to think that this value is not renormalised by higher loops ($L > 2$) although we do not have an explanation for this.
- The next possible value of k is $k = 6$. From (4.14) this corresponds to a genus-three ($h = 3$) string theory term. This is a logarithmically divergent ($m = 0$) contribution to the two-loop ($L = 2$) $D^{12} R^4$ term in (4.4). In this case we know of no reason why the value of this coefficient should be protected against contributions from higher values of L .

A plausible generalization of this argument would suggest that only terms of the form $D^{6h-6-2n} R^{4+n}$ and other terms of the same dimension can appear in the M-theory effective action after taking the eleven-dimensional limit $R_{11} \rightarrow \infty$, as was suggested in [6].

5. Reduction to lower dimensions

In the preceding sections we studied general features of L -loop Feynman diagrams of four-graviton scattering in eleven-dimensional supergravity compactified on a two-torus and the

consequences, via duality, for type IIA and IIB string theory compactified on a circle to nine dimensions. The Feynman diagrams beyond $L = 2$ are extremely complicated and their detailed structure was not used in our discussion, other than the fact the leading term in the low energy expansion behaves as $D^{2\beta_L} R^4$, where $\beta_1 = 0$, $\beta_2 = 2$ and $\beta_L \geq 2$ for $L > 2$. Obviously, since eleven-dimensional supergravity is not renormalisable it does not, by itself, give a well-defined quantum theory. In particular, short-distance properties smuggled into the momentum cut-off can only be determined with additional input. In our discussion the input consisted of the requirement that the toroidally compactified theory should be equivalent to string theory via the usual duality considerations. As a consequence, we found a number of interesting constraints on both the IIA and IIB theories, as well as consistency conditions on higher derivative terms in the eleven-dimensional theory itself.

Among these constraints was a strong non-renormalisation condition on the type IIA derivative expansion. In section 4.1 we found that the ten-dimensional II string perturbation theory amplitude at genus h has a low energy limit that begins with a term of the form $D^{2h} R^4$. Gratifyingly, this has also been verified, at least up to $h = 5$, directly in string perturbation theory in [13] (where the $D^{2k} R^4$ terms with $k < 6$ are viewed as analogues of ‘F-terms’ within the Berkovits formalism). This means that type IIA supergravity — the low energy limit of ten-dimensional IIA string theory — is more finite than might have been expected. In order for the leading h -loop behaviour to be $D^{2h} R^4$ there must be cancellations between the many Feynman diagrams that contribute to the h -loop amplitude of ten-dimensional IIA supergravity that extend the cancellations that arise at one and two loops and result in an extra factor of D^2 for every additional loop.

More explicitly, the Feynman diagram expansion of ten-dimensional IIA supergravity can be organized as a loop expansion in powers of the string coupling, e^{ϕ_A} , and so we have the leading contribution, $S_{(10)}^{(3+h)}$ at any genus h of the form (for $h > 1$)

$$\begin{aligned} S_{(10)}^{(3+h)} &= l_s^{8h-8} \Lambda_{10}^{6h-6} \int d^{10}x e^{2(h-1)\phi_A} \sqrt{-g_A} D^{2h} R^4 \\ &= \kappa_{10}^{2(h-1)} \Lambda_{10}^{6h-6} \int d^{10}x \sqrt{-g_A} D^{2h} R^4 \end{aligned} \quad (5.1)$$

which is analogous to the eleven-dimensional L -loop term in (2.10) but with $L \rightarrow h$, $\beta_L \rightarrow h$ and a cut-off Λ_{10} (and $w = 0$). The second line gives the expression as an expansion in the ten-dimensional Newton constant $\kappa_{10}^2 = l_{10}^8$, where the ten-dimensional Planck scale is related to the string scale by $l_{10} = e^{\phi_A/4} l_s$. Although the counterterms that cancel the divergences in this expression are undetermined the first line has the same form as the precise expression for the h -loop contribution to the $D^{2h} R^4$ term of ten-dimensional IIA string theory given by (4.5).

In order to understand the connection with our starting point in eleven dimensions we can refer (5.1) back to our earlier M-theory units using $R_{11} = l_s \exp(\phi_A)$ and $l_P = l_s g_s^{1/3}$, which gives

$$S_{(11)}^{(3+h)} = \frac{1}{R_{11}^3} (l_P \Lambda_{10})^{6(h-1)} (l_P R_{11}^{-1})^{3(h-1)} \int d^{11}x \sqrt{-G} (R_{11}^2 D^2)^h R^4. \quad (5.2)$$

If we relate the eleven-dimensional and ten-dimensional cut-offs by $\Lambda^3 = \Lambda_{10}^2/R_{11}$, we see that (5.2) has the form of the compactification of the finite part of the one-loop ($L = 1$) amplitude of the eleven-dimensional theory on a circle, expanded in powers of S , T and U (equation (4.7) with $L = 1$, $\beta_L = 0$, $w = 3$ and $v = h$ and a particular choice of counterterm). This shows that the particular relation between the string parameters and the M-theory parameters makes the perturbation expansion in eleven dimensions look very different from the one in lower dimensions.

Now consider the IIA supergravity after dimensional reduction to $d < 10$ dimensions. This can be implemented, for example, by compactifying the genus- h string theory amplitude (with its overall factor of s^h) on a $(10 - d)$ -torus of scale r and taking the low energy limit $r \rightarrow 0$ with $\alpha'/r \rightarrow 0$ (with the momenta and polarizations of the external gravitons in the non-compact directions). Then, for $h > 1$, simple dimensional analysis shows that (5.1) becomes

$$\begin{aligned} S_{(d)}^{(3+h)} &= l_s^{(d-2)(h-1)} \Lambda_d^{(d-4)h-6} \int d^d x e^{2(h-1)\phi_A^{(d)}} \sqrt{-g_A^{(d)}} D^{2h} R^4 \\ &= \kappa_{(d)}^{2(h-1)} \Lambda_d^{(d-4)h-6} \int d^d x \sqrt{-g_A^{(d)}} D^{2h} R^4. \end{aligned} \quad (5.3)$$

where $\phi_A^{(d)}$ is the d -dimensional dilaton and Λ_d the cut-off parameter of d -dimensional supergravity. It should be emphasized that (5.3) is a schematic representation of the low energy limit of the four-graviton amplitude that is relevant when the power of Λ_d is positive, indicating the presence of ultraviolet divergences. In obtaining (5.3) we have assumed that inverse powers of D^2 (i.e., powers of $1/s$, $1/t$ or $1/u$ in the amplitude) do not arise in the process of taking the low-energy limit of the toroidally compactified string amplitude, so that the power of Λ_d is not increased. With this assumption, we see from (5.3) that ultraviolet divergences are absent in dimensions satisfying

$$d < 4 + \frac{6}{h}, \quad (5.4)$$

for $h > 1$ (while $d < 8$ for $h = 1$).⁸ If this bound is indeed satisfied it means that ultraviolet divergences are absent to all orders in four or less dimensions,⁹ so it seems possible that four-dimensional $N = 8$ supergravity is ultraviolet finite.

It is interesting that the condition (5.4) is precisely the same condition as for maximally extended supersymmetric Yang-Mills theory ($N = 4$ Yang-Mills in $d = 4$ dimensions) given in [8] (and reproduced in a superspace formulation in [19]). This is quite remarkable since the pattern of potential divergences in the two theories is very different. To begin with, pure four-dimensional Yang-Mills theory is renormalisable by power counting in four dimensions.

⁸When the bound (5.4) is satisfied, the negative power of Λ_d in (5.3) is replaced by an expression involving inverse powers of s , t , or u , together with logarithms, associated with infrared effects and leading to infrared divergences when $d \leq 4$. This structure of the low energy limit of string theory loop amplitudes is explicitly illustrated at one loop ($h = 1$) in [18] and at two loops in [8, 20–22].

⁹In the earliest version of this paper we inadvertently stated that (5.3) might allow logarithmic ultraviolet divergences when $d = 4$, which it manifestly does not.

In the maximally supersymmetric extension the low-energy one-loop four-gluon amplitude is proportional to F^4 , while all higher loops are proportional to $D^2 F^4$ with no extra powers of D beyond two loops. Given these facts one can easily obtain the divergence bound (5.4). This striking similarity between super Yang-Mills and supergravity is presumably connected to the relation between open-string and closed-string theory known since the earliest days of string theory. This was exploited a long time ago [23] in order to obtain closed-string tree amplitudes from open-string tree amplitudes in an efficient manner, and subsequently [8] has proved to be of great interest in deciphering the structure of supergravity perturbation theory beyond tree level. A number of further fascinating correspondences between the structure of $N = 8$ supergravity amplitudes and the amplitudes of $N = 4$ super-Yang-Mills have since been discovered [24–27] which suggest that $N = 8$ might be less ultraviolet divergent than expected.

However, we should emphasize an important point which was overlooked in earlier versions of this paper. In string theory compactified on a torus there are not only ‘perturbative’ string states but also ‘non-perturbative’ Dp -branes and Neveu-Schwarz branes wrapped on the torus as well as Kaluza-Klein charges and Kaluza-Klein monopoles. The low energy limit that is relevant for describing the Feynman diagrams of quantum gravity (in which κ_4 is held fixed) keeps only the massless states in the perturbative sector. However, it is straightforward to see that an infinite subset of the non-perturbative states *necessarily* also becomes massless. This could have a profound effect on the nature of the low-energy limit irrespective of whether the perturbative loops are or are not UV finite.¹⁰

Although we have only considered the four-graviton amplitude, it is quite plausible that several features that we have discussed also hold more generally. In particular, since unitarity, along with supersymmetry, should interrelate all n -particle amplitudes it seems likely that the ultra-violet finiteness properties of A_4 will extend to the complete S-matrix. This is also supported by the fact that the present power-counting analysis can be extended in a straightforward way to other terms of the same dimension.

The considerations of this paper have been based on compactifying eleven-dimensional supergravity on a two-torus, although there should be generalizations to compactifications on higher-dimensional tori or more complicated compact spaces. The richer set of dualities should, in principle, lead to further consistency constraints on the string theory derivative expansion, but there are a number of unresolved issues concerning the rôle of $M2$ -brane (and $M5$ -brane) instantons that raise new difficulties for such generalizations.

To summarize, our circle of arguments imposed interconnected consistency conditions on eleven-dimensional supergravity and ten-dimensional string theory in its low energy limit. We should stress that in order to establish the validity of our use of perturbative approximations to the eleven-dimensional theory it would be necessary to develop a much better understanding of the full equivalence between M-theory and string theory.

Nevertheless, it is striking that these arguments suggest that the ultraviolet divergences of maximally extended supergravity could be milder than might have been anticipated and may even be absent in four dimensions.

¹⁰Similar observations have been made by H. Ooguri and J.H. Schwarz (private communication).

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A. Properties of tree-level and one-loop string theory

A.1 Tree-level four graviton scattering in type II string theory

In this appendix we will review the known coefficients of the higher derivative interactions that are contained in the tree-level and one-loop string perturbation theory expressions. Whereas the tree-level terms are easily obtained to all orders in α' , the one-loop coefficients are only known to low orders. The amplitude has the form [28–30],

$$A_4^{(2)} = \kappa_{10}^2 \hat{K} e^{-2\phi} T(s, t, u), \quad (\text{A.1})$$

where $2\kappa_{10}^2 = (2\pi)^7 \alpha'^4$ and $T(s, t, u)$ is given by

$$\begin{aligned} T &= \frac{64}{l_s^6 stu} \frac{\Gamma(1 - \frac{\alpha'}{4}s) \Gamma(1 - \frac{\alpha'}{4}t) \Gamma(1 - \frac{\alpha'}{4}u)}{\Gamma(1 + \frac{\alpha'}{4}s) \Gamma(1 + \frac{\alpha'}{4}t) \Gamma(1 + \frac{\alpha'}{4}u)} \\ &= \frac{64}{l_s^6 stu} \exp \left(\sum_{n=1}^{\infty} \frac{2\zeta(2n+1)}{2n+1} \frac{\alpha'^{2n+1}}{4^{2n+1}} (s^{2n+1} + t^{2n+1} + u^{2n+1}) \right). \end{aligned} \quad (\text{A.2})$$

Define

$$\sigma_n = \left(\frac{\alpha'}{4} \right)^n (s^n + t^n + u^n)$$

It is easy to see that all σ_n can be written in terms of σ_2 and σ_3 as follows [11]

$$\sigma_n = n \sum_{2p+3q=n} \frac{(p+q-1)!}{p!q!} \left(\frac{\sigma_2}{2} \right)^p \left(\frac{\sigma_3}{3} \right)^q. \quad (\text{A.3})$$

In terms of σ_2 , σ_3 , the low energy expansion of the amplitude reads

$$T = \frac{3}{\sigma_3} + A(\sigma_2, \sigma_3), \quad (\text{A.4})$$

with

$$\begin{aligned}
A(\sigma_2, \sigma_3) &= \sum_{p,q=0}^{\infty} T_{(p,q)} \sigma_2^p \sigma_3^q \\
&= 2\zeta(3) + \zeta(5)\sigma_2 + \frac{2}{3}\zeta(3)^2\sigma_3 + \frac{1}{2}\zeta(7)\sigma_2^2 + \frac{2}{3}\zeta(3)\zeta(5)\sigma_2\sigma_3 \\
&\quad + \frac{1}{4}\zeta(9)\sigma_2^3 + \frac{2}{27}(2\zeta(3)^3 + \zeta(9))\sigma_3^2 + \frac{1}{6}(2\zeta(3)\zeta(7) + \zeta(5)^2)\sigma_2^2\sigma_3 \\
&\quad + \frac{1}{8}\zeta(11)\sigma_2^4 + \frac{1}{9}(2\zeta(3)^2\zeta(5) + \zeta(11))\sigma_2\sigma_3^2 + \cdots.
\end{aligned} \tag{A.5}$$

The number of kinematical structures appearing at each order $D^{2k}R^4$ is given by the number of ways k decomposes as the sum of a multiple of 2 and a multiple of 3, $k = 2p + 3q$ (so that $\sigma_2^p \sigma_3^q$ corresponds to the order $s^k R^4$).

A.2 One-loop four-graviton scattering in type II string theory

The one-loop four graviton scattering is given by an integral over the complex positions ν_i ($i = 1, 2, 3, 4$) of four vertex operators on a toroidal world-sheet with complex structure τ , which is to be integrated over the fundamental domain [28],

$$A_4^{one-loop} = \frac{\kappa_{10}^4}{2^5 \pi^6 \alpha'^4} R^4 \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \int_T \prod_{i=1}^4 \frac{d^2\nu_i}{\tau_2} (\chi_{12}\chi_{34})^{\alpha's} (\chi_{14}\chi_{23})^{\alpha't} (\chi_{13}\chi_{24})^{\alpha'u}, \tag{A.6}$$

where χ_{ij} is the scalar Green function between the points ν_i and ν_j on the world-sheet torus. The low-energy expansion of this amplitude can be evaluated order by order in α' using a diagrammatic method described in [11]. Technical difficulties arise at order $\alpha'^4 s^4$ due to the presence of logarithmic massless threshold singularities. This expansion is discussed in detail in [12], where the expansion is carried out up to and including order $\alpha'^4 s^4$ giving

$$A_4^{one-loop} = \frac{\kappa_{10}^4}{2^5 \pi^6 \alpha'^4} R^4 \frac{2\zeta(2)}{\pi} \left(1 + 0 \cdot \sigma_2 + \frac{\zeta(3)}{3} \sigma_3 + 0 \cdot \sigma_2^2 + O(s^5) \right). \tag{A.7}$$

It is notable that the coefficients of the $D^4 R^4$ and $D^8 R^4$ terms vanish.

B. Ten-dimensional thresholds from nine dimensions

In this appendix we provide some details of how the nonanalytic terms of the ten-dimensional IIA theory arise by resumming and infinite series of terms in nine dimensions and taking the limit $r_A \rightarrow \infty$. The two examples discussed in the main text are (i) the one-loop box diagram (in figure 1) that gives $s \ln(-\alpha' s)$, and (ii) the triangle diagram that contains a one-loop counterterm (in figure 4), which contributes to $s^4 \ln(-\alpha' s)$.

(i) The one-loop φ^3 scalar field theory box integral compactified on a two-torus from eleven to nine dimensions is the sum of three terms, $I_{\text{Box}}(S, T) + I_{\text{Box}}(T, U) + I_{\text{Box}}(U, S)$,

containing threshold singularities in (S, T) , (T, U) and (U, S) , respectively. The function $I_{\text{Box}}(S, T)$ is given by [6, 7]

$$I_{\text{Box}}(S, T) = \frac{2\pi^9}{l_P^2 \mathcal{V}} \int_0^\infty \frac{dt}{t^{\frac{3}{2}}} \int_{\mathcal{T}_{ST}} \prod_{r=1}^3 dw_r \sum_{m_I=(m_1, m_2)} e^{-G^{IJ} m_I m_J t - Q(S, T) t} \quad (\text{B.1})$$

where $\mathcal{T}_{ST} = \{1 \geq w_1 \geq w_2 \geq w_3 \geq 0\}$, and $Q(S, T) = -S w_1(w_2 - w_3) - T(w_1 - w_2)(1 - w_3)$ where $S = -(k_1 + k_2)^2$, $T = -(k_1 + k_3)^2$ and $U = -(k_1 + k_4)^2$. For compactification on a square torus we have

$$G^{IJ} m_I m_J = \left(\frac{m_1}{R_{10}} \right)^2 + \left(\frac{m_2}{R_{11}} \right)^2. \quad (\text{B.2})$$

In the zero Kaluza-Klein sector $m_1 = m_2 = 0$ this integral has the $(-S)^{1/2}$ non-analytic behavior of the scalar field theory box diagram in nine-dimensions

$$I_{\text{Box}}^0(S, T) \sim \int_{\mathcal{T}_{ST}} Q(S, T)^{\frac{1}{2}}. \quad (\text{B.3})$$

In [6, 7] it was shown that the derivative expansion of the nine-dimensional expression for the box diagram is given by

$$\begin{aligned} I_{\text{Box}}(s, t) = & 2\zeta(3) e^{-2\phi^A} + \frac{2\pi^2}{3r_A^2} + \frac{2\pi^2}{3} - 8\pi^2 r_A^{-1} (l_s^2 \hat{D}^2)^{\frac{1}{2}} \\ & + 8\pi^{\frac{3}{2}} \sum_{n \geq 2} \frac{\Gamma(n - \frac{1}{2})}{n!} \zeta(2n - 1) r_A^{2n-2} (-l_s^2 \hat{D}^2)^n \\ & + 8\pi^2 \sum_{n \geq 2} \frac{\Gamma(n - 1)}{n!} \zeta(2n - 2) e^{2(n-1)\phi^A} (-l_s^2 \hat{D}^2)^n \\ & + \text{non-perturbative} \end{aligned} \quad (\text{B.4})$$

where $\hat{D}^{2n} = \int_{\mathcal{T}_{ST}} Q(s, t)^n$. The infinite series of powers of e^{ϕ^A} in the last line is just the series of terms in (4.5). The powers series in powers of r_A is a series that sums to give the thresholds due to massive Kaluza-Klein intermediate states,

$$\begin{aligned} r_A^{-1} (l_s^2 \hat{D}^2)^{\frac{1}{2}} R^4 - \pi^{-\frac{1}{2}} \sum_{k=2}^{\infty} \frac{\Gamma(k - \frac{1}{2})}{k!} \zeta(2k - 1) r_A^{2(k-1)} (-l_s^2 \hat{D}^2)^k R^4 = \\ = r_A^{-1} R^4 \int_{\mathcal{T}_{ST}} \sum_{n \in \mathbb{Z}} \left(\frac{n^2}{r_A^2} + l_s^2 Q(s, t) \right)^{\frac{1}{2}}. \end{aligned} \quad (\text{B.5})$$

In [7] it was shown that the series on the right-hand side of (B.5) sums to give the $s \ln(-\alpha' s)$ threshold term in the ten-dimensional $r_A = R_{10} \sqrt{R_{11}} \rightarrow \infty$ limit. This can also be seen directly from the expression for $I_{\text{Box}}(S, T)$, as follows. In the the limit of decompactification to the ten-dimensional IIA theory the scalar box integral is dominated

by the sector with $m_2 = 0$ and becomes

$$\begin{aligned} I_{\text{Box}}(S, T) &\rightarrow \frac{2\pi^9}{R_{11}^{\frac{1}{2}}} \int_0^\infty \frac{dt}{t^{\frac{3}{2}}} \int_{\mathcal{T}_{ST}} \prod_{r=1}^3 dw_r \frac{1}{r_A} \sum_{m_1 \in \mathbb{Z}} e^{-(m_1/r_A)^2 R_{11}^2 t - Q(S, T) t} \\ &\rightarrow \frac{2\pi^{\frac{11}{2}}}{R_{11}} \int_0^\infty \frac{dt}{t^2} \int_{\mathcal{T}_{ST}} \prod_{r=1}^3 dw_r e^{-Q(S, T) t} . \end{aligned} \quad (\text{B.6})$$

The last expression has the $s \ln(-\alpha' s)$ non-analytic behavior of the scalar box diagram in ten-dimensions,

$$I_{\text{Box}}^0(S, T) \sim \int_{\mathcal{T}_{ST}} Q(S, T) \ln(Q(S, T)) . \quad (\text{B.7})$$

(ii) The second example comes from considering the scalar triangle diagram (figure 4) compactified to nine dimensions. This is given by

$$I_{\text{Triang}}(S) = \frac{\pi^{\frac{9}{2}}}{l_P^2 \mathcal{V}} \sum_{m_I=(m_1, m_2)} \int_0^\infty \frac{dt}{t^{\frac{5}{2}}} \int_0^1 dw_2 \int_0^{w_2} dw_1 e^{-G^{IJ} m_I m_J t - Q(S) t} , \quad (\text{B.8})$$

where $Q(S) = -S(1 - w_2)(w_2 - w_1)$. This can be expanded in an infinite power series in S , which is given by

$$\begin{aligned} I_{\text{Triang}}(S) &= \frac{2\pi^3}{5} \hat{\Lambda}^5 + \frac{3\pi}{2} \zeta(5) e^{-2\phi^A} + 2\zeta(4) \frac{e^{2\phi^A}}{r_A^4} - \frac{4\pi^5}{3} (l_s^2 \hat{D}^2)^{\frac{3}{2}} \\ &\quad + 2\pi^{\frac{9}{2}} \sum_{n \geq 1} \frac{\Gamma(n - \frac{3}{2})}{n!} \zeta(2n - 3) r_A^{2n-4} (-l_s^2 \hat{D}^2)^n \\ &\quad + 4\pi^5 \sum_{n \geq 1} \frac{\Gamma(n - 2)}{n!} \zeta(2n - 4) e^{2(n-1)\phi^A} (-l_s^2 \hat{D}^2)^n \\ &\quad + \text{non-perturbative} \end{aligned} \quad (\text{B.9})$$

where $\hat{D}^{2n} = \int_{1 \geq w_2 \geq w_1 \geq 0} Q(s)^n$. The second line is the infinite series of positive powers of r_A in (4.11) that sum to give the nine-dimensional Kaluza-Klein thresholds. The series of higher genus terms in the third line of (B.9) corresponds to terms that are increasing powers of R_{11} . These are the terms that resum in the limit $R_{11} \rightarrow \infty$ to reproduce the $(-S)^{3/2}$ threshold term of the eleven-dimensional theory.

The fact that the sum of Kaluza-Klein thresholds gives the ten-dimensional threshold behaviour can be seen by directly analyzing $I_{\text{Triang}}(S)$ (B.8) in the $r_A \rightarrow \infty$ limit. The decompactification limit to ten-dimensional perturbative superstring gives

$$\begin{aligned} I_{\text{Triang}}(S) &\rightarrow \frac{\pi^{\frac{9}{2}}}{l_P^2 R_{11}^{\frac{1}{2}}} \int_0^\infty \frac{dt}{t^{\frac{5}{2}}} \int_0^1 dw_2 \int_0^{w_2} dw_1 \frac{1}{r_A} \sum_{m_1 \in \mathbb{Z}} e^{-(m_1/r_A)^2 R_{11}^2 t - Q(S) t} \\ &\rightarrow \frac{\pi^5}{l_P^2 R_{11}} \int_0^\infty \frac{dt}{t^3} \int_0^1 dw_2 \int_0^{w_2} dw_1 e^{-Q(S) t} \\ &\sim \frac{\pi^5}{l_P^2 R_{11}} \int_0^1 dw_2 \int_0^{w_2} dw_1 Q(S)^2 \ln(Q(S)) , \end{aligned} \quad (\text{B.10})$$

which has the $S^2 \ln(-S)$ behaviour needed for obtaining the $S^4 \ln(-S)$ non-analytic contributions at one and two loops in string theory in ten dimensions.

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